O(2) Model in $d=2$ and

Berezinskii – Kosterlitz-Thouless (BKT) Transition.

As hinted above, the case $d=2$, $n=2$ of the O(n) model in d-dimensions is rather special. This is partly because one is right at the lower critical dimension but more importantly, topological defects in the ordered phase (vortices) play a crucial role.

The model is $H = -K \sum_{\langle ij \rangle} \cos(\theta_i - \theta_j)$ where $i,j \in$ say, the square lattice. Lets first proceed in a simple minded way and assume that at low T (i.e. $K \gg 1$), one may simply expand the cosine to the lowest order :

$$H \simeq K \sum_{\langle ij \rangle} \frac{(\theta_i - \theta_j)^2}{2}$$

$$\simeq \frac{K}{2} \int d^2 r \, [\nabla \theta(r)]^2$$

where we have written down a continuum version in the second line.

An important point to note here (with hindsight) is that since θ is an angular variable $(\theta \equiv \theta + 2\pi)$, one cannot just rescale θ without changing its periodicity. For example, one might want to define $\theta' = \sqrt{K}\,\theta$, so that $H \simeq \frac{1}{2}\int (\nabla\theta')^2 \, d^2r'$.

This is fine but one then needs to be mindful that $\theta' \equiv \theta' + 2\pi\sqrt{K}$. Therefore, the coefficient of the kinetic energy term is <u>not</u> a reduntant variable, unlike the Ising model. Since it's easier to remember $\theta \equiv \theta + 2\pi$, we will retain K as in $H \simeq \frac{K}{2}\int (\nabla\theta)^2 \, d^2r$.

Secondly, we notice that K is dimensionless. Therefore, at the level of this gaussian approximation, K is a marginal variable, $\frac{dK}{dl} = 0$. This was already hinted at in our discussion of the $O(n)$ model in $d = 2 + \varepsilon$ above.

Assuming that this Gaussian approximation is valid at low enough temperatures, the correlation fn of the physical spin $\vec{S} = (\cos\theta, \sin\theta)$ are given by

$$\langle \vec{S}(r) \cdot \vec{S}(0) \rangle = \langle \cos\theta(r) \cos\theta(0) + \sin\theta(r) \sin\theta(0) \rangle$$

$$= \langle \cos(\theta(r) - \theta(0)) \rangle$$

$$= \mathrm{Re} \langle e^{i(\theta(r) - \theta(0))} \rangle$$

$$= e^{-\frac{1}{2}\langle (\theta(r) - \theta(0))^2 \rangle}$$

where we have used the fact that the cumulant expansion is exact for a Gaussian theory.

Now, $\frac{1}{2}\langle (\theta(r) - \theta(0))^2 \rangle = \langle \theta^2(0) \rangle - \langle \theta(0)\, \theta(r) \rangle$

$$= \frac{1}{K} \int \frac{d^2k}{(2\pi)^2} \frac{(1 - e^{i\vec{k}\cdot\vec{r}})}{k^2} \simeq \frac{2\pi}{K(2\pi)^2} \int_{r^{-1}}^{a^{-1}} \frac{k\, dk}{k^2}$$

where the lower limit is set by the observation that when $k \ll r^{-1}$, the integrand is essentially zero. The upper limit is set to a^{-1} where 'a' is the lattice spacing.

$$\Rightarrow \quad \frac{1}{2}\left\langle [\theta(r) - \theta(0)]^2 \right\rangle = \frac{1}{2\pi K} \log\left(\frac{r}{a}\right).$$

$$\Rightarrow \quad \langle \vec{S}(\vec{r}) \cdot \vec{S}(\vec{0}) \rangle \simeq e^{-\frac{1}{2\pi K}\log\left(\frac{r}{a}\right)}$$

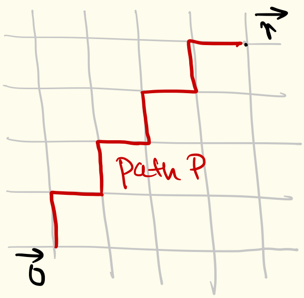
$$= \left(\frac{a}{r}\right)^{\frac{1}{2\pi K}}$$

This result is quite remarkable: not only do correlations exhibit a power-law behaviour <u>without</u> needing to <u>fine-tune</u> any parameter, the exponent defining the power-law, $\eta = \frac{1}{2\pi K}$ varies <u>continuously</u> with K and therefore, is <u>not</u> universal, unlike the exponents we encountered for the Ising criticality (or for that matter for any $O(n)$ model in $d=3$), which were universal and required tuning atleast one parameter. This is again a consequence of the fact that K is exactly marginal in this Gaussian theory, and that this theory is gapless (i.e. a term θ^2 in H is not allowed).

In contrast, at very high temperature ($K \ll 1$), it is easy to show that the correlation f^{n}s decay exponentially. The argument is same as in any other local model:

$$\langle e^{i(\theta(r) - \theta(0))} \rangle \simeq \frac{\mathrm{Tr} \prod_{\langle ij \rangle} [1 + K \cos(\theta_i - \theta_j)] \, e^{i(\theta(r) - \theta(0))}}{\mathrm{Tr} \prod_{\langle ij \rangle} [1 + K \cos(\theta_i - \theta_j)]}$$

Since $\mathrm{Tr}\, e^{i\theta} = 0$, at the leading order, the non-zero contribution in the numerator comes from the term which involves the product $\prod_{i \in P} K \cos(\theta_i - \theta_j)$ where P is the shortest path connecting $\vec{0}$ to $\vec{r}$.

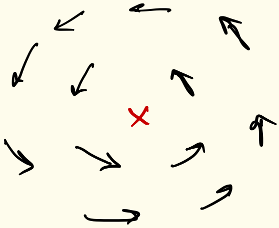


$$\Rightarrow \langle e^{i(\theta(r) - \theta(0))} \rangle \sim K^r$$

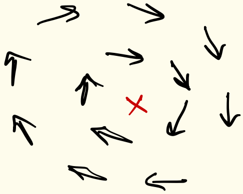
$$\sim e^{-r \log(1/K)}.$$

Therefore, there must exist a temperature at which the low-T power law behaviour ends. The simplest possibility is that for $K > K_c$, one obtains the aforementioned power law behavior ($\langle s(r) \cdot s(0) \rangle \sim (1/r)^{1/2\pi K}$ and for

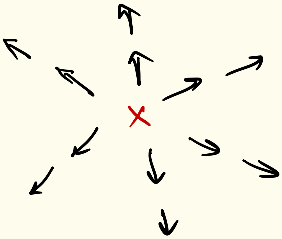
$K < K_c$, one obtains $\langle S(r)\cdot S(0)\rangle \sim e^{-r/\xi}$.
This turns out to be indeed the case.
The physical reason for the onset of the exponential behavior is that the periodic nature of θ becomes important as K decreases.
As fluctuations of θ increase, one obtains configurations where θ winds by 2π around some point. Such configurations are called vortices. The 'math reason' for the existence of vortices is that the order parameter is a two-comp. vector : $(\cos\theta, \sin\theta)$ that lives on a circle, and along a closed path in real-space, one can define a winding number of θ which is ofcourse an integer. Since integers can't change continuosly, an isolated vortex is therefore a point-like stable particle, akin to a charge in electromagnetism that cannot be destroyed or created in isolation. Lets consider a few vortex configurations.



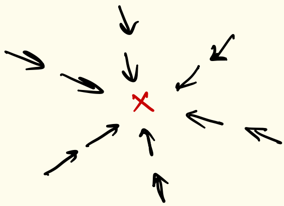
vorticity = +1



vorticity = +1



vorticity = +1



vorticity = +1

$$\text{vorticity} = \frac{1}{2\pi} \oint_C \vec{\nabla}\theta \cdot d\vec{\ell}$$

Contour C encloses the vortex core.

Homework: plot config. with vorticity = -1, $+2$, -2.

When do vortices become important? Lets consider the contribution to free energy from a single vortex (this is somewhat analogous to our discussion of free energy contribution from a domain wall in an Ising model).

Order parameter $\theta(\vec{r})$ for a vortex configuration,

$$\theta(r, \varphi) = n\varphi \Rightarrow \vec{\nabla}\theta = \frac{n}{r}\hat{\varphi} \quad \text{where}$$

n is the winding number ('vorticity').

$$\Rightarrow \text{Energy cost} = \frac{K}{2}\int (\nabla\theta)^2 d^2r = \frac{K}{2}\int_a^L \frac{n^2}{r^2} d^2r$$

where a is the short-distance cutoff $\sim$ vortex core size, below which the coarse-grained expressions such as $\theta_i - \theta_j \sim \nabla\theta$ do not make sense. L is the total system size.

$$\Rightarrow \text{Energy cost} = \frac{K}{2} n^2 \times 2\pi \int_a^L \frac{dr}{r}$$

$$= \pi n^2 K \log\left(\frac{L}{a}\right)$$

Entropy of a single vortex $\simeq$

$\log$ (# of locations where vortex can be placed)

$$\approx \log\left(\left(\frac{L}{a}\right)^2\right) = 2\log\left(\frac{L}{a}\right).$$

Thus, the free energy contribution $= \Delta E - T\Delta S$

$$= (\pi n^2 K - 2)\log\left(\frac{L}{a}\right).$$

Thus, a single vortex with minimum possible non-zero vorticity ($n^2 = 1$) is not suppressed when $K < \frac{2}{\pi}$, and therefore $K_c > \frac{2}{\pi}$.

One issue with this argument is that a single vortex costs energy $\log(L)$ which is divergent, and therefore should never exist.

The cure is that a vortex-antivortex separated by distance l_0 cost energy $\pi K \log\left(\frac{l_0}{a}\right)$, and in the above argument one may replace $L \to l_0$ and obtain the same estimate as above for a lower bound on K_c.